Sect. 18.9.2 Alternative Method

A reader, Luc Longtin, came up with an alternative method for evaluating the many paths transition amplitude of Sect. 18.9.2, pgs. 504-505. Those lacking familiarity with the convolution integral approach should find this method easier to understand than that used in the text. However, it is a bit longer.

Evaluating the Amplitude Section, pg. 504

To evaluate (18-50), we make repeated use of (18-46). We first perform the integration over the x_1 variable; that is, the rightmost integral in (18-50). We have the following:

$$\ldots\int_{-\infty}^{+\infty} e^{\frac{im}{2\hbar(\Delta t)}(x_2-x_1)^2}\, e^{\frac{im}{2\hbar(\Delta t)}(x_1-x_i)^2}\,dx_1 = \ldots\int_{-\infty}^{+\infty} e^{\frac{im}{2\hbar(\Delta t)}\left(x_2^2-2x_2x_1+x_1^2\right)}\, e^{\frac{im}{2\hbar(\Delta t)}\left(x_1^2-2x_ix_1+x_i^2\right)}\,dx_1$$

$$=\ldots\int_{-\infty}^{+\infty} e^{\frac{im}{2\hbar(\Delta t)}\left(x_2^2+x_i^2\right)}\, e^{-\frac{im}{\hbar(\Delta t)}(x_2+x_i)x_1}\, e^{\frac{im}{\hbar(\Delta t)}\left(x_1^2\right)}\,dx_1 = \ldots e^{\frac{im}{2\hbar(\Delta t)}\left(x_2^2+x_i^2\right)}\int_{-\infty}^{+\infty} e^{\frac{im}{\hbar(\Delta t)}\left(x_1^2\right)}\, e^{-\frac{im}{\hbar(\Delta t)}(x_2+x_i)x_1}\,dx_1$$

The integral on the RHS is seen to be of form (18-46) with:

$$a=-\frac{im}{\hbar(\Delta t)} \qquad \text{and} \qquad b=-\frac{im}{\hbar(\Delta t)}(x_2+x_i)$$

Using the result of (18-46), we can write:

$$=\ldots e^{\frac{im}{2\hbar(\Delta t)}\left(x_2^2+x_i^2\right)}\sqrt{\frac{\pi\,\hbar(\Delta t)}{-im}}\; e^{\left(-\frac{im}{\hbar(\Delta t)}(x_2+x_i)\right)^2\Big/4\left(-\frac{im}{\hbar(\Delta t)}\right)} = \ldots\sqrt{\frac{i\pi\,\hbar(\Delta t)}{m}}\; e^{\frac{im}{2\hbar(\Delta t)}\left(x_2^2+x_i^2\right)}\, e^{-\frac{im}{4\hbar(\Delta t)}(x_2+x_i)^2}$$

$$=\ldots\sqrt{\frac{i\pi\,\hbar(\Delta t)}{m}}\; e^{\frac{im}{2\hbar(\Delta t)}\left[\left(x_2^2+x_i^2\right)-\frac{(x_2+x_i)^2}{2}\right]} = \ldots\sqrt{\frac{i\pi\,\hbar(\Delta t)}{m}}\; e^{\frac{im}{2\hbar(\Delta t)}\frac{(x_2-x_i)^2}{2}} = \ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\; e^{\frac{im}{2\hbar(\Delta t)}\frac{(x_2-x_i)^2}{2}}$$

We now proceed to the second integration in (18-50), over the x_2 variable; that is:

$$=\ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\int_{-\infty}^{+\infty} e^{\frac{im}{2\hbar(\Delta t)}(x_3-x_2)^2}\, e^{\frac{im}{2\hbar(\Delta t)}\frac{(x_2-x_i)^2}{2}}\,dx_2$$

$$=\ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\int_{-\infty}^{+\infty} e^{\frac{im}{2\hbar(\Delta t)}\left(x_3^2-2x_3x_2+x_2^2\right)}\, e^{\frac{im}{2\hbar(\Delta t)}\frac{\left(x_2^2-2x_ix_2+x_i^2\right)}{2}}\,dx_2$$

$$=\ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\int_{-\infty}^{+\infty} e^{\frac{im}{2\hbar(\Delta t)}\left(x_3^2+\frac{x_i^2}{2}\right)}\, e^{\frac{im}{2\hbar(\Delta t)}\left(\frac{3x_2^2}{2}\right)}\, e^{-\frac{im}{2\hbar(\Delta t)}(2x_3+x_i)x_2}\,dx_2$$

$$=\ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\; e^{\frac{im}{2\hbar(\Delta t)}\left(x_3^2+\frac{x_i^2}{2}\right)}\int_{-\infty}^{+\infty} e^{\frac{im}{2\hbar(\Delta t)}\left(\frac{3x_2^2}{2}\right)}\, e^{-\frac{im}{2\hbar(\Delta t)}(2x_3+x_i)x_2}\,dx_2$$

The integral on the RHS is seen to be of form (18-46) with:

$$a = -\frac{im}{\hbar(\Delta t)}\frac{3}{4} \qquad \text{and} \qquad b = -\frac{im}{\hbar(\Delta t)}\left(x_3 + \frac{x_i}{2}\right)$$

Using the result of (18-46), we can write:

$$= \ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\, e^{\frac{im}{2\hbar(\Delta t)}\left(x_3^2+\frac{x_i^2}{2}\right)} \sqrt{\frac{\pi\,\hbar(\Delta t)}{-im}\frac{4}{3}}\, e^{\left[-\frac{im}{\hbar(\Delta t)}\left(x_3+\frac{x_i}{2}\right)\right]^2 \Big/ 4\left(-\frac{im}{\hbar(\Delta t)}\frac{3}{4}\right)}$$

$$= \ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\, e^{\frac{im}{2\hbar(\Delta t)}\left(x_3^2+\frac{x_i^2}{2}\right)} \sqrt{\frac{ih(\Delta t)}{m}\frac{2}{3}}\, e^{-\frac{im}{\hbar(\Delta t)}\frac{1}{3}\left(x_3+\frac{x_i}{2}\right)^2}$$

$$= \ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\sqrt{\frac{ih(\Delta t)}{m}\frac{2}{3}}\, e^{\frac{im}{2\hbar(\Delta t)}\left(x_3^2+\frac{x_i^2}{2}-\frac{2}{3}\left(x_3+\frac{x_i}{2}\right)^2\right)}$$

$$= \ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\sqrt{\frac{ih(\Delta t)}{m}\frac{2}{3}}\, e^{\frac{im}{2\hbar(\Delta t)}\left(\frac{x_3^2-2x_3x_i+x_i^2}{3}\right)} = \ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\sqrt{\frac{ih(\Delta t)}{m}\frac{2}{3}}\, e^{\frac{im}{2\hbar(\Delta t)}\frac{(x_3-x_i)^2}{3}}$$

We see a pattern emerging: for the total of *n* integrations in (18-50), we will get a product of *n* square roots and an exponential factor. We have found that exponential factor to be:

after one integration, $e^{\frac{im}{2\hbar(\Delta t)}\frac{(x_2-x_i)^2}{2}}$, after two integrations, $e^{\frac{im}{2\hbar(\Delta t)}\frac{(x_3-x_i)^2}{3}}$.

We thus expect the exponential factor after *n* integrations to be $e^{\frac{im}{2\hbar(\Delta t)}\frac{(x_{n+1}-x_i)^2}{n+1}} = e^{\frac{im}{2\hbar T}(x_f-x_i)^2}$.

In the last equality, we have used the fact that $x_f = x_{n+1}$ and $T = (n+1)\Delta t$.

As to the product of square roots, we might guess the following result after three integrations:

$$\ldots\sqrt{\frac{ih(\Delta t)}{m}\frac{1}{2}}\sqrt{\frac{ih(\Delta t)}{m}\frac{2}{3}}\sqrt{\frac{ih(\Delta t)}{m}\frac{3}{4}} = \ldots\left(\frac{ih(\Delta t)}{m}\right)^{3/2}\sqrt{\frac{1}{4}} \quad \text{(our guess; validated in problem)}$$

To determine the actual result for the square root factors, we must perform the third integration, over the x_3 variable. This is left as an exercise for you to do, and confirm our guess.

Having these results, we easily extrapolate to write the end result for the *n* integrations of (18-50). We thus have:

$$U(x_i, x_f; T) \approx C\left(\frac{ih(\Delta t)}{m}\right)^{n/2}\sqrt{\frac{1}{n+1}}\, e^{\frac{im}{2\hbar T}(x_f-x_i)^2} = C\left(\frac{ih(\Delta t)}{m}\right)^{n/2}\sqrt{\frac{\Delta t}{T}}\, e^{\frac{im}{2\hbar T}(x_f-x_i)^2}$$

$$U(x_i, x_f; T) \approx C\left(\frac{ih(\Delta t)}{m}\right)^{n/2}\sqrt{\frac{ih(\Delta t)}{m}}\sqrt{\frac{m}{ihT}}\, e^{\frac{im}{2\hbar T}(x_f-x_i)^2}$$

$$U(x_i, x_f; T) \approx C\left(\frac{ih(\Delta t)}{m}\right)^{(n+1)/2} \sqrt{\frac{m}{ihT}}\, e^{\frac{im}{2\hbar T}(x_f - x_i)^2} = C\left(\frac{i2\pi\hbar(\Delta t)}{m}\right)^{(n+1)/2} \sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{im}{2\hbar T}(x_f - x_i)^2},$$

where $n + 1 = N$ of Chap. 18, (18-59). n = number of integrations.

From this result, we see that the dependence on x_i, x_f and T, is *wholly* contained in the last two factors on the RHS, namely the square root and the exponential factor. The constant C can be determined by comparison with (18-47).

$$U(x_i, x_f; T) = \sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}\frac{m}{2T}(x_f - x_i)^2} \quad \text{from Schroedinger approach} \tag{18-47}$$

to get

$$C = \left(\frac{m}{ih(\Delta t)}\right)^{(n+1)/2} .$$

Thus, we arrive at (18-61),

$$\left|U(x_i, x_f; T)\right|^2 = \frac{m}{2\pi\hbar T} \tag{18-61}$$